Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 4, pp. 593–597, 1995

0021–8928/95 $24.00+0.00

0021–8928(95)00069-0

SOME SPECIAL FEATURES OF PROBLEMS OF THE STABILITY AND VIBRATIONS OF RECTANGULAR PLATES†

N. V. BANICHUK and A. Yu. ISHLINSKII

Moscow

(*Received* 25 *August* 1994)

Problems of the stability and free vibrations of elastic rectangular plates are considered in the development of an earlier investigation [1] (also, see [2]). It is shown that, in the limit, there is no transition of the two-dimensional solutions into one-dimensional "beam" solutions and an estimate is given of the difference between the limiting characteristics and the corresponding quantities obtained assuming that the deformation of the plates is cylindrical.

In problems of the elastic stability, vibrations and static bending of infinitely long rectangular plates which are clamped along the long sides and are loaded with forces which remain constant along the length, it is usually assumed that the deflection function is a cylindrical surface. On this basis, the solution of the initial two-dimensional problem is replaced by the solution of the one-dimensional problem for a narrow transverse strip or beam with a reduced Young's modulus $E' = E/(1-\nu^2)$, where ν is Poisson's ratio. However, this alternative, which enables one to take account of the resistance to transverse contraction of the material, which accompanies cylindrical flexure, is not completely rigorous and does not provide an adequate description of the deformation of long rectangular plates. The true deflection function in the direction of the length and width has curvatures of different signs and the ratio of these curvatures on the free edges of the plates remains constant for an arbitrary length-to-width ratio. Taking into account the substantially two-dimensional character of the behaviour of the modes of the stability loss and vibrations of elastic plates with distant free edges, we would expect the limiting solutions of two-dimensional problems, when the length of the plates tend to infinity, to differ from the "beam" solutions of the corresponding one-dimensional problems. This special feature when taking the limit was first observed [1] in connection with problems of the stability of elastic rectangular plates.

1. We consider the problem of the loss of elastic stability of a plate, the two opposite edges of which of length $2b$ are hinged while the other two edges of length l are free (Fig. 1). Distributed compressive forces of magnitude p are applied to the hinged edges. Finding the critical magnitude of the load under which the plate loses stability and is buckled reduces to finding the least eigenvalue p and the eigenfunction corresponding to it (the mode of stability loss) $w = w(x, y)$ from the solution of the following boundary-value problem for the eigenvalues

$$D\Delta^2 w + p w_{xx} = 0 \quad (D = Eh^3/[12(1-\nu^2)]) \tag{1.1}$$

$$w = 0, \quad w_{xx} = 0 \text{ when } x = 0, l, \quad -b \leqslant y \leqslant b \tag{1.2}$$

$$w_{yy} + \nu w_{xx} = 0, \quad w_{yyy} + (2-\nu) w_{xxy} = 0 \quad \text{when} \quad y = \pm b, \quad 0 \leqslant x \leqslant l \tag{1.3}$$

For the longitudinal flexure equation (1.1) with the support boundary conditions (1.2) and the boundary conditions when there are no moments and cutting forces on the free edges (1.3). Here, D is the cylindrical stiffness of the plate and h is the thickness.

It should be noted that a one-dimensional "cylindrical" distribution of the deflections $w = \varphi(x)$ can never be an eigenfunction of problem (1.1)–(1.3) for any finite values of p and b/l. In fact, it follows from the boundary condition when there are no moments on the free edges (1.3) $\nu w_{xx} - w_{yy}$ that $\varphi_{xx} = 0$ when $y = \pm l$ and $0 \leqslant x \leqslant l$. Taking into account the support boundary conditions $\varphi = \varphi_{xx} = 0$ when $x = 0$ and $x = l$, we obtain $\varphi(x) \equiv 0$, $0 \leqslant x \leqslant l$.

†*Prikl. Mat. Mekh.* Vol. 59, No. 4, pp. 620–625, 1995.

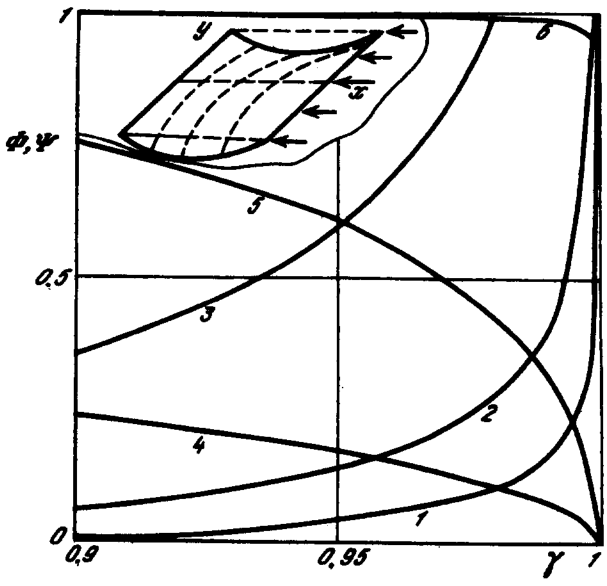

Fig. 1.

It may be assumed, however, that the critical force for stability loss p and the distribution of the deflections corresponding to it when $l/b \to 0$ will differ by as small an amount as is desired from the known one-dimensional solution

$$p_{cr} = \pi^2 D/l^2, \quad w = C\sin(\pi x/l) \tag{1.4}$$

of the problem of the stability loss of an infinitely long panel with the formation of a cylindrical form of buckled surface (C is an arbitrary non-zero constant).

In order to examine how rigorous this assertion is, we shall carry out calculations which are only slightly different from those presented in [1, 2]. The distribution of the deflections in problem (1.1)–(1.3) is given by the expression

$$w = f(\eta)\sin(\pi x/l), \quad \eta = \pi y/l \tag{1.5}$$

where $f(\eta)$ is the eigenfunction which corresponds to the least eigenvalue

$$\mu = \gamma^2 = pl^2/(\pi^2 D) \tag{1.6}$$

of the following boundary-value problem

$$f'''' - 2f'' + (1-\mu)f = 0 \tag{1.7}$$

$$f'' - \nu f = 0, \quad f''' - (2-\nu)f' = 0 \text{ when } \eta = \pm\pi b/l \tag{1.8}$$

When $\gamma \leqslant 1$, a solution of (1.5), (1.7) which is symmetric with respect to the x-axis can be represented in the form

$$w = (A\operatorname{ch}(\sqrt{1+\gamma}\,\eta) + B\operatorname{ch}(\sqrt{1-\gamma}\,\eta))\sin(\pi x/l) \tag{1.9}$$

where A and B are arbitrary constants. The condition for a non-trivial solution to exist in the form (1.9), as follows from (1.5), (1.8) and (1.9) ($\eta = \pi b/l$) leads to the transcendental equation [3]

$$\Phi(\gamma, \varepsilon) \equiv \operatorname{th}\frac{\sqrt{1-\gamma}}{\varepsilon}\operatorname{cth}\frac{\sqrt{1+\gamma}}{\varepsilon} = \frac{\sqrt{1+\gamma}}{\sqrt{1-\gamma}}\frac{(\gamma+\nu-1)^2}{(\gamma-\nu+1)^2} \equiv \Psi(\gamma, \nu) \tag{1.10}$$

for determining the eigenvalue $\mu = \gamma^2$ for various values of the parameter $\varepsilon = l/(\pi b)$.

Let us investigate the properties of the functions $\Phi(\gamma, \varepsilon)$ and $\Psi(\gamma, \nu)$ when $0 \leqslant x \leqslant 1$. We note that the range of critical values of p, which exceed the Euler force $p_e = \pi^2 D(1-\nu^2)/l^2$ for the stability loss of a clamped hinged bar of length l and are not greater than the force $p = \pi^2 D/l^2$ for buckling into a cylindrical shape, corresponds to the interval

$$1 - \nu^2 < \mu = \gamma^2 < 1 \tag{1.11}$$

As γ increases from zero to unity, the function $\Phi(\gamma, \varepsilon)$ decreases continuously and monotonically from 1 to 0. The function $\Psi(\gamma, \nu)$ decreases from 1 to 0 in the interval $0 < \gamma < 1 - \nu$ while it increases monotonically in the interval $1 - \nu < \gamma < 1$ and takes values as large as are desired when $\gamma \to 1$.

Plots of the function $\Psi(\gamma, \nu)$ when $\nu = 0.2, 0.3$ and 0.5 are shown in Fig. 1 by the lines with the numbers 1, 2 and 3. The functions $\Phi(\gamma, \varepsilon)$ when $l/b = 10, 1, 0.1$ ($\varepsilon = l/(\pi b)$) are shown by the lines with the numbers 4, 5 and 6.

The value of $\gamma = \gamma_0 \in [1 - \nu, 1]$ for which $\Psi(\gamma_0, \nu) = 1$ is of special interest. On solving the corresponding equation, we obtain

$$\gamma_0^2 = (1-\nu)[3\nu - 1 + 2(1 - 2\nu(1-\nu))^{1/2}] \tag{1.12}$$

The value of γ_0, found earlier in [1], turns out to be close to unity. The value of $\mu = \gamma_0^2$ will be equal to 0.9994, 0.9962 and 0.9571 for values of Poisson's ratio $\nu = 0.2$, 0.3 and 0.5, respectively. Assuming that $l \gg b$ and that $\varepsilon = l/(\pi b)$ is large, we have the approximate equation

$$\Psi = ((1-\gamma)/(1+\gamma))^{1/2}$$

and Eq. (1.10) admits of the solution $\gamma_e^2 = 1 - \nu^2$. This solution corresponds to a narrow strip (bar) which is supported and compressed at its far ends and leads to the Euler value of the force for stability loss

$$p_e = \pi^2 D\gamma_e^2/l^2 = \pi^2 Eh^3/(12l^2)$$

It follows from the above treatment and the properties of the functions $\Phi(\gamma, \varepsilon)$, $\Psi(\gamma, \nu)$ which have been noted that the roots of Eq. (1.10), when $0 < \varepsilon < \infty$, lie in the interval, lie in the interval $\gamma_e \leqslant \gamma < \gamma_0$. In the limit of infinitely long plates, that is when $\varepsilon \to 0$, we have $\gamma \to \gamma_0 \neq 1$, and the corresponding mode of stability loss (1.9) does not turn out to be cylindrical. The largest difference between the magnitude of the critical load which leads to stability loss of an infinitely long plate and the corresponding value, obtained assuming a distribution of the deflections over a cylindrical surface, occurs when $\nu = 0.5$, that is, in the case of an absolutely incompressible material.

We will now investigate the possibility of modes of buckling which are antisymmetric about the x-axis

$$w = f(\eta)\sin(\pi x/l), \quad f = A\,\mathrm{sh}(\sqrt{1+\gamma}\,\eta) + B\,\mathrm{sh}(\sqrt{1-\gamma}\,\eta) \tag{1.13}$$

when $\gamma \leqslant 1$.

Using expression (1.13) for f and the boundary conditions on the free edge of the plate (1.8), we obtain the following transcendental equation for determining the quantity

$$\Phi(\gamma, \varepsilon) = \Psi^{-1}(\gamma, \nu) \tag{1.14}$$

In the segment $0 \leqslant \gamma \leqslant 1$ being considered, Eq. (1.14) has two roots γ_1 ($\gamma_0 < \gamma_1 < 1$) and γ_2 ($\gamma_2 = 1$) for arbitrary values of ν and the parameter ε which characterizes the elongation of the plate. Hence, the quantity $\gamma_2 = 1$, which corresponds to the critical value of the stability loss parameter in the problem of cylindrical bending, turns out to be an eigenvalue of the problem being considered of stability loss in the case of a plate with a finite length-to-width ratio. However, the mode of stability loss corresponding to this eigenvalue is not symmetric and the eigenvalue itself is not the least eigenvalue $\gamma_0 < \gamma_2$ and it is not the magnitude of the critical load.

2. We will now consider the problem of the natural harmonic transverse vibrations of a rectangular plate. Using the earlier notation and assuming that the conditions for the clamping of the edges of the plate to be same as the previous conditions, we write the equation for the deflection amplitude function

$$D\Delta^2 w - \rho h\omega^2 w = 0 \tag{2.1}$$

where ω is the frequency of natural vibrations. We shall seek the function w in the form of the expression

$$w = f(\eta)\sin(m\pi x/l), \quad \eta = \pi y/l \tag{2.2}$$

which satisfies the boundary conditions on the hinged edges of the plate ($x = 0, l$) and describes the distributions of the deflections with m half waves in the direction of the x-axis. Here, the function $f(\eta)$ must satisfy the following equation when $-\beta < \eta < \beta$ ($\beta = \pi b/l$) and the conditions on the free edges

$$\frac{1}{m^4}f'''' - \frac{2}{m^2}f'' + (1-\gamma^2)f = 0, \quad \gamma^2 = \frac{\rho h l^4\omega^2}{m^4\pi^4 D} \tag{2.3}$$

$$f'' - m^2\nu f = 0, \quad f''' - (2-\nu)m^2 f' = 0 \quad \text{when} \quad \eta = \pm\beta \tag{2.4}$$

A solution of (2.3), when $\gamma^2 \leqslant 1$, which is symmetric about the x-axis can be written in the form

$$f = A\operatorname{ch}(m\eta\sqrt{1+\gamma}) + B\operatorname{ch}(m\eta\sqrt{1-\gamma}) \tag{2.5}$$

with the arbitrary constants A and B. The condition for a non-trivial solution of the form of (2.5) to exist, which satisfies the boundary conditions (2.4), leads to a transcendental equation which reduces to (1.1) if we put $m\beta = 1/\varepsilon$. In this case, we can use the properties of the solutions of the equation $\Phi(\gamma, \varepsilon) = \Psi(\gamma, \nu)$ which have been mentioned. For instance, when $l/b \to \infty$, that is, for $\varepsilon = 1/(m\beta) = 1/(m\pi b) \to \infty$, the quantity $\gamma^2 \to 1 - \nu^2$ and the frequency of the vibrations of the plate asymptotically approach the corresponding frequency

$$\omega^2 = \frac{m^4\pi^4 D}{\rho h l^4}\gamma^2 = \frac{m^4\pi^4 EI}{\rho h l^4}$$

of the transverse vibrations of a beam which is clamped along its edges. As the ratio l/b ($\varepsilon \to 0$) becomes smaller, the quantity γ^2 tends to the value γ^2_0 which is determined using formula (1.12).

Note that the limiting value $\gamma = \gamma_0$ is independent of the number of half waves m in the case of the mode of vibration of the plate being considered. Hence, when $l/b \to 0$, the frequency of the symmetric vibrations of the plate do not tend to the frequencies which are obtained from the solution of the problem assuming a cylindrical form of the mode of vibrations.

Let us now consider the behaviour of the solution of Eq. (2.3) which are asymmetric about the axis

$$f = A\operatorname{sh}(m\eta\sqrt{1+\gamma}) + B\operatorname{sh}(m\eta\sqrt{1-\gamma}) \tag{2.6}$$

and correspond to the assumption that $\gamma^2 \leqslant 1$. In this case, non-trivial solutions of the form of (2.6) which satisfy the boundary conditions (2.4) are possible if γ is a root of an equation which reduces to a form identical with (1.14) if we put $m\beta = 1/\varepsilon$. From this, when account is taken of the above properties of the functions Φ and Ψ and the graphical data shown in Fig. 1, we conclude that two roots γ_1 and γ_2 exist in the case of Eq. (1.14) and, moreover, $\gamma_0 < \gamma_1 < \gamma_2 = 1$. Here, it should also be noted that, in the case of natural vibrations with any number of half waves m, the quantity $\gamma_2 = 1$ determines the frequencies of the asymmetric modes which are identical with the frequencies of the symmetric vibrations which are found from the solution of one-dimensional problems assuming a cylindrical form of the deflection distributions.

The search for the symmetric and antisymmetric natural modes and the corresponding natural frequencies of the free vibrations when $\gamma^2 > 1$ reduces to finding the non-trivial solutions of Eq. (2.3) of the form

$$\begin{gathered} f = A\operatorname{ch}(m\eta\sqrt{\gamma+1}) + B\cos(m\eta\sqrt{\gamma-1}) \\ f = C\operatorname{sh}(m\eta\sqrt{\gamma+1}) + K\sin(m\eta\sqrt{\gamma-1}) \end{gathered} \tag{2.7}$$

where A, B, C and K are arbitrary constants. The values of γ, which ensure the existence of non-trivial solutions in the form of (2.7) are, respectively, the roots of the equations

$$\Lambda(\gamma,\varepsilon)=-\Psi(\gamma,\nu),\quad \Lambda(\gamma,\varepsilon)=\Psi^{-1}(\gamma,\nu) \tag{2.8}$$
$$\Lambda\equiv \operatorname{tg}(\sqrt{\gamma-1}/\varepsilon)\operatorname{cth}(\sqrt{\gamma+1}/\varepsilon)$$

In the case when $\gamma > 1$, the set of roots γ and the frequencies corresponding to them are found from Eqs (2.8).

3. It follows from the above treatment that the property of a passage to the limit noted in [1] in a problem of the elastic stability of plates with free edges is also characteristic in the case of dynamical problems on the natural vibrations of the plates with analogous boundary conditions. In this case, the limiting difference between the solution of the two-dimensional problem and the corresponding one-dimensional "cylindrical" solution with respect to the magnitude of the first eigenvalue (the critical force for stability loss and the fundamental frequency) increases as Poisson's ratio increases and is greatest in the case of a plate made of an incompressible material. The same difference in the magnitudes of the eigenvalues also holds in the case of the higher symmetric vibrational modes and the forms of stability loss. Moreover, the magnitude of the relative divergence of the eigenvalues being considered is independent of the number of half waves in the direction of the unclamped sides of the plate.

The behaviour of the asymmetric vibrational modes and asymmetric equilibrium forms of longitudinal flexure of a plate are of interest. For instance, the least eigenvalue which corresponds to an asymmetric form of buckling in the stability problem turns out to be insignificantly greater than the critical force for the stability loss of the plate. The next eigenvalue in magnitude $\gamma^2 = 1$, which corresponds to asymmetric forms of the problem of elastic stability, turns out to be identical with the value of the critical force for stability loss in the corresponding one-dimensional problem of buckling in the form of a cylindrical surface.

The spectrum of the eigenvalues and characteristic frequencies of the asymmetric vibrations possesses similar singularities. In particular, asymmetric vibrations with $\gamma = 1$ correspond to the characteristic frequencies of the symmetric "cylindrical" vibrations.

From a mathematical point of view, the appearance of singularities is closely related to the asymptotic properties of the solution in the case of long plates with free edges. The singularities, on passing to the limit when $l/b \to 0$, lie in the fact that it is performed in the case of an equation with a small parameter accompanying the highest derivative, the investigation of which is associated with the treatment of boundary-layer solutions. The problems discussed in this paper are of interest in this context as an example of problems of this kind which admit of an analytic solution.

It would seem that the special features of the asymptotic behaviour of the solutions of the problems of the stability and vibrations of elastic plates which have been noted, are related to the characteristic "inaccuracies" of the "classical" theory of plates which is used. However, it should be noted here that the discrepancies which have been revealed, as follows from the above calculations, are not due to the combined Kirchhoff boundary condition for a cutting force but to the condition that the bending moment is equal to zero. Actually, the condition that there is no bending moment on the free edge precludes the appearance of eigenfunctions in the form of cylindrical surfaces.

We make the following remark with regard to the applicability of Saint Venant's principle to the treatments carried out in this note and the apparent paradox. One-dimensional cylindrical solutions of problems of the elastic stability of plates and panels are characterized by non-zero bending moments at the infinitely distant edges. The non-one-dimensional solutions for elongated rectangular plates satisfy the condition that the bending moment is equal to zero on the free edges for any elongations (aspect ratio) of a plate as if they earn a zero resultant to infinity. Therefore, the above-mentioned solutions do not refer to one and the same class of solutions with statically equivalent resulting force characteristics.

REFERENCES

1. ISHLINSKII A. Yu., On a limiting process in the theory of the stability of elastic rectangular plates. *Dokl. Akad. Nauk SSSR* **95**, 3, 477–479, 1954.
2. ISHLINSKII A. Yu., *Applied Problems in Mechanics*, Vol. 2. Nauka, Moscow, 1986.
3. DONNELL L. G., *Beams, Plates and Shells*. Nauka, Moscow, 1982.

Translated by E.L.S.